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Crack propagation, arrest and statistics in heterogeneous materials

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Abstract. We investigate theoretically statistics and thermally activated dynamics of crack nucleation and propagation in a two-dimensional heterogeneous material containing *quenched randomly distributed* defects. We consider a crack tip dynamics accounting for dissipation, thermal noise and the random forces arising from the elastic interactions of the crack opening with the defects. The equation of motion is based on the generalized Griffith criterion and the dynamic energy release rate and gives rise to Langevin-type stochastic dynamics in a quenched disordered potential. For different types of quenched random forces, which are characterized (a) by the range of elastic interactions with the crack tip and (b) the range of correlations between defects, we derive a number of static and dynamic quantities characterizing crack propagation in heterogeneous materials both at zero temperature and in the presence of thermal activation. In the absence of thermal fluctuations we obtain the nucleation and propagation probabilities, typical arrest lengths, the distribution of crack lengths and of critical forces. For thermally activated crack propagation we calculate the mean time to fracture. Depending on the range of elastic interactions between crack tip and frozen defects, heterogeneous material exhibits brittle or ductile fracture. We find that aggregations of defects generating long-range interaction forces (e.g. clouds of dislocations) lead to anomalously slow creep of the crack tip or even to its complete arrest. We demonstrate that heterogeneous materials with frozen defects contain a large number of arrested microcracks and that their fracture toughness is enhanced to the experimentally accessible timescales.

Keywords: disordered systems (theory), fracture (theory), extreme value problems

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1. Introduction

The fracture mechanism of different materials depends crucially on the structure of the material and is usually termed brittle or ductile, depending on the amount of dissipated energy in the process of crack propagation [1]. Homogeneous crystalline materials show fast brittle fracture as first explained by Griffith [2], whereas homogeneous amorphous materials exhibit slow ductile fracture dominated by plastic deformation at the crack tip [1, 3]. In brittle fracture there is essentially no plastic deformation at the tip of a propagating crack and, thus, the dissipated energy remains small, such that much of the elastic energy can be converted into kinetic energy, giving rise to fast cracks. In ductile fracture, on the other hand, the material around the crack tip undergoes considerable plastic deformation, and the resulting motion of dislocations provides an effective mechanism to dissipate energy into the surrounding solid. Therefore, the energy needed for the crack advance is relatively large, and cracks in ductile materials are typically much slower than in brittle materials. Ductile materials are often amorphous, i.e. they contain a large number of mobile defects, such as dislocations, whereas defect-free crystals are brittle.

However, the realistic situation is more complicated because materials are *disordered* and contain quenched heterogeneities, i.e. *immobile defects* such as dislocation pileups, inclusions or defective molecular bonds [1, 4, 5]. Such quenched defects are present already before a crack starts propagating and are not mobile. Therefore, they cannot directly contribute to energy dissipation. In this paper, we specifically consider a random array of such heterogeneities, which is embedded in an otherwise defect-free crystal as schematically shown in figure 1. On the one hand, such arrays of quenched defects are

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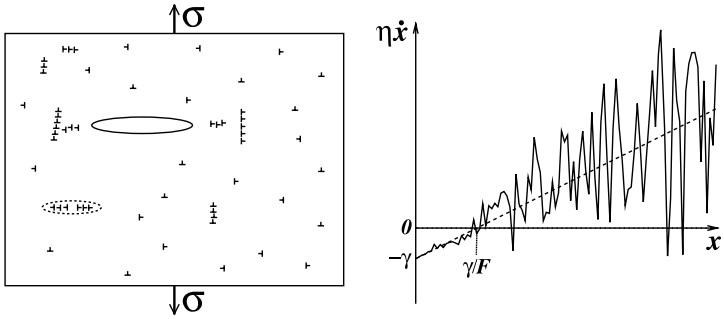

Figure 1. Left: sketch of an arrested crack in a random array of frozen dislocations; the dashed crack indicates a favorable region for crack nucleation. Right: typical realization of the effective force $-\gamma+Fx+f_{\mathrm{d}}(x)$, see equation (10), acting on the crack tip in a random array of frozen dislocations ($\delta = 3$ in equation (4)).

not mobile and, thus, do not provide an effective way of dissipating energy as in a ductile material. On the other hand, such defects interact with the crack via elastic deformations and, therefore, exert quenched random forces onto the crack tip. This raises two kinds of questions: first, whether such quenched defects favor or obstruct crack nucleation and propagation and, second, whether the resulting fracture mechanism could still be characterized as brittle or becomes ductile.

It is intuitively plausible that crack nucleation is favored by defects as the crack can choose a favorable nucleation site, where the interaction energy with the surrounding defects lowers the nucleation barrier for subcritical cracks. However, after having chosen a favorable nucleation site the crack has to propagate through the array of heterogeneities in a direction, which is essentially determined by the applied force. Having chosen its nucleation site the crack position is not optimized for the subsequent propagation process. Therefore, it appears much less clear how this propagation process is affected by random heterogeneities and, in general, one rather expects that subsequent crack propagation is slowed down in a random array of defects. This raises the question whether frozen defects enhance or degrade the fracture toughness of a material. In order to answer this question the crack dynamics also has to be considered. An anomalously slow crack propagation can prevent fracture on the experimental timescale. The inspiring work [6] addressed the question of the disorder-stimulated nucleation of critical cracks. In this paper, we first reconsider this zero-temperature problem of calculating the fracture probability of a disordered medium under an external load for two more general classes of quenched defects. Then we discuss static crack length distributions and the extreme value statistics of critical forces. The second part of the paper goes beyond the static analysis and focuses on the *dynamics* of cracks in heterogeneous materials and the statistics of fracture times. A short account of some of the results discussed in this paper regarding the crack dynamics has already appeared as a letter [7]. Quenched immobile defects interact with the crack tip and give rise to energy barriers for crack propagation. We consider crack equations of motion which include thermal noise and allow it to overcome such barriers by thermal activation. This process gives rise to a slow crack dynamics. It is well known that thermal activation plays an important role for crack nucleation [8] and

propagation [9]. Slow thermally activated crack dynamics has been previously discussed for energy barriers arising from lattice trapping effects [10, 11] and additional disorder effects in random fuse models [12, 13]. Whereas in these works only *finite* energy barriers slow down crack propagation, we show that for materials containing quenched defects the long-range elastic interaction with the crack tip can give rise to much higher energy barriers diverging with the crack length, which can slow down crack motion much more effectively and lead to crack arrest or anomalously slow crack dynamics with a zero mean velocity.

The other question is whether the fracture of a crystal resembles brittle or ductile fracture in the presence of defects or heterogeneities, which are quenched. In the presence of a sufficiently small concentration of quenched defects the fracture process is expected to remain brittle in general. If the *mobility* of defects increases, as has been observed in Si for dislocations [14] or stacking faults, i.e. dislocation pileups [15] above a certain temperature, a transition to ductile fracture can take place. In heterogeneous two-dimensional random spring [16] or random fuse [17, 18] network models with *quenched* random spring or fuse breaking thresholds, on the other hand, defects are absent but a similar brittle-to-ductile transition has been found numerically if the distribution of spring or fuse breaking thresholds is broadened. In these networks, microcracks tend to be generated for broad spring or fuse breaking threshold distributions, and fracture proceeds by coalescence of such microcracks rather than crack propagation in the ductile regime. In the ductile or tough regime the fraction of damaged material prior to unstable growth of a crack is of the order of unity, whereas it remains small in the brittle regime. Using analytical arguments it has been shown in [19] that such a transition is absent and fracture is brittle in a one-dimensional heterogeneous fiber composite. More recent numerical work on random fuse and random spring networks shows that the fracture is indeed brittle, with a strong damage localization before crack propagation but the damage precursors exhibit behavior similar to percolation [20]. In this paper, we will focus on the effects of *immobile* defects on the propagation of a single crack, i.e. we neglect effects from the interaction between cracks and the coalescence of microcracks. Our results suggest that a transition from a brittle to an effectively ductile fracture mechanism can arise as a function of the scaling properties of the disorder-induced elastic forces, by which the immobile defects interact with the crack tip. For short-range disorder-induced elastic forces, a single crack shows unstable growth above a certain critical length set by the disorder strength, which is typical for a brittle material. For sufficiently long-range disorder-induced elastic forces, on the other hand, the crack becomes arrested or anomalously slow and further propagation requires an increase of the external load. Such a behavior is qualitatively similar to ductile fracture.

We restrict ourselves to the simplest situation of cracks in a thin (quasi-) two-dimensional ideally elastic plate containing a random array of heterogeneities. In two dimensions the crack front is a point—the crack tip—such that additional effects from crack front roughening are absent. Building on Griffith's concept of energy balance [2], we consider crack tip motion governed by the dynamic energy release rate [3, 21] and derive an equation of motion for the crack tip that includes dissipative and thermal forces, and position-dependent random forces acting on the crack tip due to frozen material defects. Starting from such an equation of motion for crack dynamics we can treat both statics and dynamics of cracks for different kinds of frozen heterogeneities, which are characterized

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by the scaling properties of their elastic interactions with the crack opening. We find that the fracture behavior strongly depends on these elastic interactions and introduce two generic classes of crack tip forces, namely short- and long-range correlated forces (SRCF and LRCF), which are characterized by an exponent δ. We derive all results for the general case of arbitrary δ. Following [6] we consider in particular three special cases of frozen heterogeneities: (i) a random fracture toughness from bond strength variations, (ii) random impurities giving rise to local compression of the elastic medium and (iii) randomly placed frozen dislocations interacting with the crack. The range of elastic interactions with the crack tip increases from type (i) to (ii) and (iii). We find that supercritical cracks are permanently arrested by sufficiently long-range disorder-induced elastic forces, such as those generated by frozen dislocations (iii). Considering the thermally activated dynamics of propagating cracks, we derive anomalously slow dynamics with vanishing mean velocity for all three types (i)–(iii) of disorder. Static and dynamic findings are interpreted in a unified manner in terms of the distribution of fracture times. We conclude that quenched material defects provide a very effective mechanism for slowing down crack propagation and find characteristic experimentally observable material properties such as the extreme value statistics of critical stresses, power-law distributions of crack waiting times and the existence of ensembles of arrested microcracks. For sufficiently long-range disorder-induced elastic forces we find anomalously slow or even arrested cracks. This favors a ductile material failure by alternative mechanisms such as microcrack coalescence or a slow fatigue mechanism under cyclic loading.

2. Crack tip equation of motion

Let a single planar crack extend from $-x/2$ to $+x/2$ along the x direction of a two-dimensional elastic medium of size L loaded in mode I by a uniform external stress σ. We will consider a perfectly straight crack and neglect the possibility that the crack path could deviate from a straight line because of disorder effects. In a perfectly homogeneous elastic medium, the Griffith crack energy $E(x)$ is the sum of the elastic energy gain $E_{\rm el}(x)$ and the crack surface energy $E_{\rm s}(x)$. The driving force for crack tip advance is the release of elastic energy, which is given by the *static energy release rate* $G(x)$ [1]:

$$G(x) = -\partial_x E_{\rm el} = K^2(x)/Y = \pi\sigma^2 x/2Y \equiv Fx, \tag{1}$$

where $K(x) \equiv \sigma(\pi x/2)^{1/2}$ is the *stress intensity factor* and $Y = E/(1-\nu^2)$ is Young's modulus in 2D. The energy release rate is balanced by the specific *fracture energy* γ related to the crack surface energy $E_{\rm s}$:

$$\gamma = \partial_x E_{\rm s}. \tag{2}$$

Griffith's criterion for the onset of crack growth gives $G > G_{\rm c} = \gamma$ [2], where $G_{\rm c}$ is the *critical energy release rate* that can be reached by increasing the crack length beyond its critical value:

$$x_{\rm c} = \gamma/F = 2\gamma Y/\pi\sigma^2. \tag{3}$$

Material heterogeneities are incorporated into Griffith's force balance as additional frozen random forces $f_{\rm d}(x)$. As in [6], we take Gaussian distributed random forces with zero mean $\overline{f_{\rm d}(x)} = 0$, where the overbar denotes the average over disorder, and consider

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two types of forces: short- and long-range correlated forces (SRCF and LRCF) with

$$\overline{f_{\rm d}(x)f_{\rm d}(x')} = \Delta_\delta x^\delta \delta_a(x-x') \qquad \text{(SRCF)}, \tag{4}$$

$$\overline{f_{\rm d}(x)f_{\rm d}(x')} = \frac{\delta+1}{4}\Delta_\delta x^{(\delta-1)/2}x'^{(\delta-1)/2} \qquad \text{(LRCF)}, \tag{5}$$

where a is a microscopic cutoff length, Δ_δ is the strength of the random forces proportional to the defect concentration and the exponent δ characterizes the elastic interaction between frozen defects and the crack. The corresponding disorder potential energy $E_{\rm d}(x)$ of the crack tip defined via $f_{\rm d}(x) = -\partial_x E_{\rm d}$ has correlations

$$\overline{(E_{\rm d}(x)-E_{\rm d}(x'))^2} = \frac{\Delta_\delta}{\delta+1}|x^{\delta+1}-x'^{\delta+1}| \qquad \text{(SRCF)}, \tag{6}$$

$$\overline{(E_{\rm d}(x)-E_{\rm d}(x'))^2} = \frac{\Delta_\delta}{\delta+1}(x^{(\delta+1)/2}-x'^{(\delta+1)/2})^2 \qquad \text{(LRCF)}. \tag{7}$$

For large separations $|x-x'| \gg x$, both types of random forces give rise to the same correlations $\overline{(E_{\rm d}(x)-E_{\rm d}(x'))^2} \sim \Delta_\delta|x-x'|^{\delta+1}$ of the disorder potential energy. It was shown in [6] that (i) random bonds (random fracture toughness) result in SRCF with $\delta = 0$, (ii) impurities produce SRCF with $\delta = 1$ and (iii) dislocations induce LRCF with $\delta = 3$, i.e. δ increases with the range of the elastic interaction between the crack tip and defects. In general, we expect $\delta = 5-2\alpha$ for the heterogeneity-induced power-law stress fields decaying as $\sigma \sim r^{-\alpha}$. For frozen impurities, (ii), $\sigma \sim r^{-2}$ ($\alpha = 2$), and for frozen dislocations, (iii), $\sigma \sim r^{-1}$ ($\alpha = 1$). Random surface energies, (i), are of a *finite* range corresponding to $\delta = 0$.

In the following, we will introduce two types of equation of motion for the crack tip: an overdamped dynamics of the crack tip derived from Griffith's crack energy and an equation of motion governed by the dynamic energy release rate. To derive the overdamped dynamics of the crack tip one has to include dissipative and thermal forces into the force balance. Dissipation occurs mostly near the crack tip where the energy is converted into heat by plastic deformation [21]. Thus dissipation can be described as a local viscous force $-\eta\dot{x}$ with the crack tip viscosity η. Including thermal forces $\zeta(t)$ we obtain the overdamped equation of motion of the crack tip:

$$\eta\dot{x} = G(x) - \gamma + f_{\rm d}(x) + \zeta(t) = -\gamma + Fx + f_{\rm d}(x) + \zeta(t), \tag{8}$$

where the thermal noise and viscous force are related via $\langle\zeta(t)\zeta(t')\rangle = 2\eta T\delta(t-t')$ ($\langle\zeta(t)\rangle = 0$, $k_{\rm B} \equiv 1$). A simple overdamped dynamics is appropriate for very slow cracks and has been widely used in modeling crack front roughening [22, 23]. However, it does not capture important features emerging at higher crack tip velocities such as the existence of a terminal crack velocity [3]. Then the kinetic energy of the elastic medium has to be taken into account by using the *dynamic energy release rate* $G(x,\dot{x})$ [3]:

$$G(x,\dot{x}) = A(\dot{x})G(x) \approx (1-\dot{x}/c_{\rm R})\,G(x). \tag{9}$$

In general, $A(\dot{x})$ decreases monotonically with the increasing crack tip velocity $\dot{x}$ with $A(0) = 1$ in the static limit and $A(c_{\rm R}) = 0$, where $c_{\rm R} \propto (Y/\rho)^{1/2}$ is the Rayleigh wave velocity (ρ is the mass density) [3]. The last approximation in equation (9) is a particularly simple functional form which is appropriate for most experiments and will be used in what

follows. It should also be noted that equation (9) is not valid at very high crack velocities where branching of cracks sets in [21]. In a homogeneous material the dynamic force balance $G(x,\dot{x}) = \gamma$ generalizes the Griffith's criterion and describes the energy flux into the crack tip and its subsequent conversion into the crack surface energy. Including dissipation by the viscous force $-\eta\dot{x}$, the thermal forces $\zeta(t)$, as well as the frozen random forces $f_{\rm d}(x)$ into the dynamic force balance, $-\eta\dot{x} + G(x,\dot{x}) = \gamma - f_{\rm d}(x) - \zeta(t)$, one finally arrives at the equation of motion for the energy release dynamics:

$$\eta\dot{x} = B(x)\left[-\gamma + Fx + f_{\rm d}(x) + \zeta(t)\right], \tag{10}$$

$$B(x) \equiv \left[1 + (\gamma/\eta c_{\rm R})(x/x_{\rm c})\right]^{-1} \equiv \left[1 + b/x\right]^{-1}, \tag{11}$$

which resembles the overdamped dynamics (8) with an effectively x-dependent temperature determined by the dimensionless function $B(x)$; $b \equiv \eta c_{\rm R}/F$ is the characteristic crack length. For small cracks $x \ll b$, $B(x) \approx 1$, and equation (10) reduces to the overdamped dynamics (8). For large cracks $x \gg b$; on the other hand, $B(x)$ vanishes as $B(x) \approx b/x$, giving rise to a small effective temperature.

3. Crack statistics at zero temperature

3.1. Nucleation probability

In the absence of any flaws for $f_{\rm d}(x) = 0$, microcracks can be thermally nucleated for $T > 0$ [8], but the material would be stable against crack formation at $T = 0$. However, there is a non-zero probability that cracks of the critical size $x \sim x_{\rm c} = \gamma/F$ are also 'nucleated' by the quenched disorder at $T = 0$ if the disorder energy gain can compensate for the nucleation energy $\Delta E_{\rm c} = \gamma^2/2F$ [6]. The probability $p_{\rm nucl}$ for a disorder-induced nucleation can be calculated by noting that the disorder energy $E_{\rm d}(x_{\rm c}) = \int_0^{x_{\rm c}} {\rm d}y\, f_{\rm d}(y)$ (setting $E_{\rm d}(0) \equiv 0$) is Gaussian-distributed with $\overline{E_{\rm d}^2(x_{\rm c})} \sim \Delta_\delta x_{\rm c}^{1+\delta}/(1+\delta)$, which scales in the same way for both SRCF and LRCF. The resulting nucleation probability is

$$\begin{aligned} p_{\rm nucl} &= {\rm prob}[\Delta E_{\rm c} + E_{\rm d}(x_{\rm c}) < 0] \\ &\sim {\rm e}^{-(\Delta E_{\rm c})^2/2\overline{E_{\rm d}^2(x_{\rm c})}} \sim {\rm e}^{-(1+\delta)\gamma^{3-\delta}F^{\delta-1}/8\Delta_\delta}, \end{aligned} \tag{12}$$

which increases with increasing disorder strength Δ_δ and is identical for SRCF and LRCF.

Alternatively, the nucleation probability can be obtained from the random force statistics as

$$p_{\rm nucl} = {\rm prob}[-\gamma + f_{\rm d}(x) > 0 \,{\rm for}\, 0 < x < x_{\rm c}], \tag{13}$$

which is the probability that a crack can spontaneously grow up to $x = x_{\rm c} = \gamma/F$ at $T = 0$ against the crack surface force γ, which dominates for such subcritical cracks. For SRCF the nucleation probability can be computed from the Gaussian distribution of independent random forces:

$${\rm prob}[\{f_{\rm d}(x)\}] \sim \exp\left(-\frac{1}{2\Delta_\delta}\int_0^\infty {\rm d}x\, x^{-\delta} f_{\rm d}(x)^2\right) \tag{14}$$

as

$$\ln p_{\rm nucl} = \int_0^{x_{\rm c}} \frac{{\rm d}x}{a} \ln\left[\int_\gamma^\infty {\rm d}f\, \exp\left(-\frac{a}{2\Delta_\delta}x^{-\delta}f^2\right)\right] \sim -\int_0^{x_{\rm c}} {\rm d}x \frac{1}{2\Delta_\delta}x^{-\delta}\gamma^2, \tag{15}$$

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giving

$$p_{\rm nucl} \sim {\rm e}^{-x_{\rm c}^{1-\delta}\gamma^2/\Delta_\delta} \sim {\rm e}^{-\gamma^{3-\delta}F^{\delta-1}/\Delta_\delta}, \tag{16}$$

in general agreement with (12). Using a characteristic force scale

$$F_\gamma \equiv (\Delta_\delta \gamma^{\delta-3})^{1/(\delta-1)}, \tag{17}$$

which is related to the crack surface energy per length γ, the results (12) or (16) can be written as

$$p_{\rm nucl} \sim {\rm e}^{-(F/F_\gamma)^{\delta-1}}. \tag{18}$$

Interestingly, for $\delta > 1$, the nucleation probability decreases with increasing force F. This is an effect of the small critical length $x_{\rm c}$ for large F, which does not allow for an effective disorder energy gain for $\delta > 1$.

3.2. Arrest length

After nucleation, a growing crack can get arrested by tip forces due to frozen disorder, thus preventing fracture. In order to calculate the propagation probability at zero temperature we have to analyze the competition between the driving force Fx and the random force $f_{\rm d}(x)$, which can eventually stop the crack if $Fx + f_{\rm d}(x) < 0$. The typical crack length in a heterogeneous material is set by the crack *arrest length* x^*, where the driving force on the crack is balanced by the typical stopping force from heterogeneities, $Fx^* = (\overline{f_{\rm d}(x^*)^2})^{1/2}$ (we focus on supercritical cracks $x \gg x_{\rm c}$ and neglect the crack surface energy γ).

For SRCF we use $(\overline{f_{\rm d}(x)^2})^{1/2} \sim (\Delta_\delta/a)^{1/2} x^{\delta/2}$ and find a typical length scale

$$x^* \sim \left(\frac{F^2 a}{\Delta_\delta}\right)^{1/(\delta-2)}. \tag{19}$$

For $\delta > 2$ the force equilibrium is *unstable* with respect to crack propagation and the material behaves as brittle. We find $Fx \gg f_{\rm stop}(x) \sim x^{\delta/2}$ for large $x \gg x^*$ and the stopping force is not sufficient to arrest the crack. Then the probability that the stopping force exceeds the driving force becomes exponentially small for large x. Small cracks $x < x^*$, however, are stable as the stopping force exceeds the driving force. Therefore, x^* has a similar meaning as a disorder-induced critical crack length for $\delta < 2$. The actual critical crack length is the maximum of x^* and $x_{\rm c} = \gamma/F$ for $\delta < 2$. We will assume a weak collective disorder, which has a small amplitude Δ_δ and becomes effective only by the cooperative effects of many impurities. In this limit we typically have $x^* \gg x_{\rm c}$.

For $\delta > 2$, on the other hand, the force equilibrium is *stable* with respect to crack propagation and x^* is the typical length of arrested cracks stabilized by the disorder stopping force. For $x > x^*$ the probability that the driving force exceeds the arresting force drops exponentially. As a result the probability for spontaneous fracture of a sample of size L becomes exponentially small $\sim {\rm e}^{-L}$ for $\delta > 2$ if $L \gg x^*$. This is reminiscent of a ductile fracture behavior. For the *marginal* case $\delta = 2$ the crack stability does *not* depend on the crack length but solely on the ratio $F/\Delta_2 \propto \sigma^2/Y\Delta_2$ of external driving force and disorder. In particular, the propagation probability only depends on the ratio F/Δ_2 and the system size L. Because for large sample size L, the crack samples many different random forces and eventually hits a stopping force with finite but small probability, the

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probability for spontaneous fracture of a sample of size L still becomes exponentially small $\sim e^{-L}$ also for $\delta = 2$ for large L.

For LRCF the result is slightly different. Then we have a typical stopping force $(\overline{f_d(x)^2})^{1/2} \sim {\Delta_\delta}^{1/2} x^{(\delta-1)/2}$, which leads to an arrest length

$$x^* \sim \left(\frac{F^2}{\Delta_\delta}\right)^{1/(\delta-3)} \tag{20}$$

and a shift by one in the behavior in the parameter δ such that the critical δ for the transition from brittle to ductile behavior becomes $\delta = 3$: for $\delta < 3$ cracks are unstable with respect to propagation, for $\delta > 3$ they are stable, and the case of frozen dislocations $\delta = 3$ corresponds to the marginal case where stability depends on the parameter only depending on the ratio F/Δ_3 and the system size L. But for LRCF also a long crack only samples one random force configuration such that the probability of spontaneous fracture of a sample of size L does not depend on L for LRCF.

3.3. Propagation probability

Following [6] we consider the probability to fracture—given a certain initial position of the crack where it has been nucleated with a length x_c. This equals the probability that a crack propagates through the sample starting with a length x_c from a certain initial point, i.e. in a certain array of random forces, and is given by the probability of violating the static force equilibrium, such that we find a positive force on the crack tip for all $x_c < x < L$:

$$p_{\text{prop}} = \text{prob}[Fx + f_d(x) > 0 \,\text{for}\, x_c < x < L]. \tag{21}$$

Whereas the nucleation probability only depends on the driving force F, the propagation probability can also depend on the finite system size L, up to which the crack has to elongate. Similar considerations have been used previously to study crack propagation [24, 25]. In these approaches it was also assumed that the crack tip deviates from a straight line. Then the crack tip can optimize its propagation direction, which gives rise to an extreme value distribution of the random forces.

For SRCF, we can calculate the propagation probability (21) using the Gaussian distribution of random forces (14). Using this we find for the probability to find crack propagation in a sample of *finite size* L:

$$\begin{aligned} \ln p_{\text{prop}} &= \int_{x_c}^{L} \frac{dx}{a} \ln\left[\int_{-F}^{\infty} d\tilde{f} \exp\left(-\frac{a}{2\Delta_\delta} x^{-\delta+2} \tilde{f}^2\right)\right] \\ &\sim \int_{x_c}^{L} \frac{dx}{a} \ln\left[1 - e^{-(x/x^*)^{-\delta+2}}\right], \end{aligned} \tag{22}$$

with x^* given by equation (19). Depending on whether $\delta < 2$ or $\delta > 2$, the last integral shows two qualitatively different behaviors. For $\delta < 2$ the exponential becomes small for $x \gg x^*$, which means that the probability of finding stable minima decreases exponentially for $x \gg x^*$. Then we can cut the x integral at $x \sim x^*$ and find

$$\ln p_{\text{prop}} \sim -\frac{x^*}{a} \qquad (\text{SRCF}, \delta < 2). \tag{23}$$

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Here we assumed that $x^* \gg x_c$, i.e. a weak collective disorder. Otherwise we have to cut the integral at x_c, which leads to the results (2.43) or (2.45) derived by Arndt and Nattermann in [6] for $\delta = 0$ and $\delta = 1$, respectively. For $\delta > 2$ we can find stopping forces with increasing probability for all $x > x^*$. Then the x integral is cut only at the system size $x \sim L$, leading to

$$\ln p_{\text{prop}} \sim \frac{L}{a} \ln \left[1 - \mathrm{e}^{-(x^*/L)^{\delta-2}}\right] \qquad (\text{SRCF},\ \delta > 2). \tag{24}$$

Therefore, for $\delta > 2$ and $L \gg x^*$ the probability for a crack to propagate through the sample is vanishing *exponentially* with the system size L; only if the characteristic stable crack length x^* approaches the sample size L can a crack propagate through the whole sample.

For $\delta < 2$, the resulting propagation probability is a finite number *independent* of L:

$$p_{\text{prop}} \sim \mathrm{e}^{-x^*/a} \sim \mathrm{e}^{-(F_a/F)^{2/(2-\delta)}} \qquad (\text{SRCF},\ \delta < 2), \tag{25}$$

with the characteristic force scale

$$F_a \equiv \left(\Delta a^{\delta-3}\right)^{1/2} \qquad (\delta \leq 2). \tag{26}$$

F_a is a small force determined by the condition $x^*(F_a) \sim a$, i.e. the force necessary to make a small crack of length $\sim a$ unstable or critical for $\delta < 2$. Such a result is typical for the brittle fracture. For $\delta > 2$, on the other hand, we find

$$p_{\text{prop}} \sim \left[1 - \mathrm{e}^{-(x^*/L)^{\delta-2}}\right]^{L/a} \sim \left[1 - \mathrm{e}^{-(F/F_L)^2}\right]^{L/a} \qquad (\text{SRCF}, \delta > 2), \tag{27}$$

with a characteristic force scale

$$F_L \equiv \left(L^{\delta-2}\frac{\Delta}{a}\right)^{1/2} \qquad (\text{SRCF}, \delta > 2), \tag{28}$$

which is a very large force determined by the condition $x^*(F_L) \sim L$. This result is characteristic for a ductile behavior because p_{prop} increases only if the stable crack of length x^* is extended through the entire sample of size L.

The marginal case $\delta = 2$ needs special consideration. For the short-range force correlator (4) with $\delta = 2$ we find instead of (22)

$$\begin{aligned} \ln p_{\text{prop}} &\sim \int_{x_c}^{L} \frac{\mathrm{d}x}{a} \ln \left[1 - \exp\left(-\frac{a}{2\Delta_2}F^2\right)\right] \\ &\simeq \frac{L}{a} \ln \left[1 - \exp\left(-\frac{a}{2\Delta_2}F^2\right)\right], \end{aligned} \tag{29}$$

resulting in

$$p_{\text{prop}} \sim \left[1 - \mathrm{e}^{-(F/F_a)^2}\right]^{L/a} \qquad (\text{SRCF},\ \delta = 2), \tag{30}$$

with $F_a \equiv (\Delta/a)^{1/2}$ as in (26). Therefore, also for $\delta = 2$ and $L \gg x_c$ the probability for a crack to propagate through the sample is vanishing *exponentially* with the system size L qualitatively similar to the case $\delta > 2$.

For LRCF, the calculation has to be done with more care because the kernel $K(x, x') \equiv \overline{f_d(x) f_d(x')}$ for the Gaussian probability distribution of random forces, as given

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by (5), has only a *single* non-zero eigenvalue as opposed to the short-range correlators which have L/a non-zero eigenvalues. This is not only a technical difficulty but reflects relevant physics: for LRCF there exists only a *single* force mode which is able to stop the crack, whereas for SRCF we find L/a potential stopping modes (one per site along the crack). Therefore, the propagation probability for $\delta \geq 3$ does not become exponentially small with increasing L/a as for the corresponding case $\delta > 2$ for SRCF. One finally finds

$$p_{\mathrm{prop}} \sim 1 - \mathrm{e}^{-(x^*/L)^{\delta-3}} \sim 1 - \mathrm{e}^{-(F/F_L)^2} \qquad \text{(LRCF)}, \tag{31}$$

with x^* given by equation (20) and the characteristic force scale

$$F_L \equiv \left(\Delta_\delta L^{\delta-3}\right)^{1/2} \qquad \text{(LRCF)} \tag{32}$$

for LRCF. For $\delta > 3$, F_L is diverging with the system size corresponding to a vanishing fracture probability

$$p_{\mathrm{prop}} \sim \left(\frac{F}{F_L}\right)^2 \sim \left(\frac{L}{x^*}\right)^{3-\delta} \approx 0 \qquad \text{for } L \gg x^* \qquad (\text{LRCF}, \delta > 3). \tag{33}$$

For $\delta < 3$, on the other hand, F_L becomes small for increasing system size and

$$p_{\mathrm{prop}} \approx 1 \qquad (\text{LRCF},\ \delta < 3), \tag{34}$$

i.e. a crack can easily propagate.

For the marginal case $\delta = 3$, which is the relevant case for frozen dislocations, $F_L = \Delta_3^{1/2}$ is independent of system size L and

$$p_{\mathrm{prop}} \sim 1 - \mathrm{e}^{-F^2/\Delta_3} \qquad (\text{LRCF},\ \delta = 3). \tag{35}$$

It is remarkable that the crack nucleation probability is

$$p_{\mathrm{nucl}} \sim \mathrm{e}^{-F^2/\Delta_3} \sim 1 - p_{\mathrm{prop}} \tag{36}$$

in this case, see (12). This shows that nucleation and propagation roughly exclude each other, leading to an enhanced stability: for $F^2 \gg \Delta_3$ cracks are not nucleated, whereas for $F^2 \ll \Delta_3$ cracks can nucleate but not propagate. Nattermann and Arndt find $\Delta_3 \sim c_{\mathrm{fd}} b_{\mathrm{fd}}^2 \sigma^2 \sim c_{\mathrm{fd}} b_{\mathrm{fd}}^2 FY$ and thus $F^2/\Delta_3 \sim \epsilon^2/c_{\mathrm{fd}} b_{\mathrm{fd}}^2$, where c_{fd} is the concentration of frozen dislocations, b_{fd} their Burgers vector and $\varepsilon = \sigma/Y$ the homogeneous strain resulting from an applied stress in the absence of a crack. As long as strains are smaller than a characteristic strain ε_{c}, $\varepsilon \ll \varepsilon_{\mathrm{c}} \equiv b_{\mathrm{fd}}/d_{\mathrm{fd}}$, where $d_{\mathrm{fd}} \sim 1/\sqrt{c_{\mathrm{fd}}}$ is the typical distance between frozen dislocations, the propagation probability (35) stays small. Estimates of Nattermann and Arndt for strong disorder in glass ($c_{\mathrm{fd}} = 10^{16}$ m^{-2} and $b_{\mathrm{fd}} = 5 \times 10^{-8}$ m) give $\varepsilon_{\mathrm{c}} = b_{\mathrm{fd}}/d_{\mathrm{fd}} \simeq 0.05$. A strain of 5% is considerable for a brittle material such as glass, which demonstrates the enhanced stability in the presence of frozen dislocations.

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3.4. Fracture probability

Spontaneous fracture at $T = 0$ requires both nucleation and propagation [6], and the probability of fracture at $T = 0$ is given by the product

$$\begin{aligned} p_{\rm frac} &= {\rm prob}[Fx - \gamma + f_{\rm d}(x) > 0 \,{\rm for}\, 0 < x < L] \\ &\simeq {\rm prob}[-\gamma + f_{\rm d}(x) > 0 \,{\rm for}\, 0 < x < x_{\rm c}] \\ &\quad \times {\rm prob}[Fx + f_{\rm d}(x) > 0 \,{\rm for}\, x_{\rm c} < x < L] \\ &= p_{\rm nucl} p_{\rm prop}. \end{aligned} \tag{37}$$

The nucleation probability $p_{\rm nucl}$ depends on F, γ and the disorder strength, as given by (12) or (18), whereas the propagation probability $p_{\rm prop}$ does not depend on γ but on the applied force F, the disorder strength, and eventually the sample size L and is given by (25), (27) and (30) for SRCF, and (31) for LRCF. For SRCF we have

$$p_{\rm frac} \sim \begin{cases} {\rm e}^{-(F/F_\gamma)^{\delta-1} - (F_a/F)^{2/(2-\delta)}} & ({\rm SRCF},\ \delta < 2) \\ {\rm e}^{-(F/F_\gamma)^{\delta-1}} \left[1 - {\rm e}^{-(F/F_L)^2}\right]^{L/a} & ({\rm SRCF},\ \delta > 2), \end{cases} \tag{38}$$

as follows from the results (25), (27) and (18). For LRCF we find

$$p_{\rm frac} \sim {\rm e}^{-(F/F_\gamma)^{\delta-1}} \left[1 - {\rm e}^{-(F/F_L)^2}\right] \qquad ({\rm LRCF}), \tag{39}$$

as follows from the results (31) and (18). For $\delta < 2$ for SRCF and $\delta < 3$ for LRCF, the fracture probability does not vanish in the limit of large L, which characterizes the brittle disordered material.

These $T = 0$ results for the fracture probability can also be rephrased in terms of the probability $p_\infty \equiv {\rm prob}[\tau_{\rm frac} = \infty]$ for a crack to have an infinite propagation or fracture time $\tau_{\rm frac}$. Because positive crack tip forces for all x imply a *finite* fracture time, we have $p_\infty = 1 - p_{\rm frac}$ at $T = 0$. For $\delta < 2$ for SRCF and $\delta < 3$ for LRCF, where fracture is brittle, we find $p_\infty < 1$, whereas $p_\infty \approx 1$ approaches 1 in the limit of large system size L for the ductile regimes $\delta > 2$ for SRCF and $\delta \geq 3$ for LRCF.

As a function of the disorder strength, the nucleation probability as given by (12) is *increasing*, showing that disorder assists in nucleation. The propagation probabilities (25), (27), (30) and (31) are all *decreasing* for increasing disorder strength, showing that propagation is opposed by disorder. Thus, there can exist an intermediate value for the disorder strength, which maximizes the fracture probability (37). For SRCF and $\delta < 2$ the maximal fracture probability is attained for $\Delta_\delta \sim a\gamma^{2-\delta} F^\delta$. For $\delta > 2$, on the other hand, the propagation probability dominates and increasing disorder strength always decreases the fracture probability. For LRCF a maximal fracture probability is only attained for $\delta \geq 3$ for $\Delta_\delta \sim \gamma^{3-\delta} F^{\delta-1}$. For $\delta = 3$ we find the previously discussed case of frozen dislocations, where nucleation and propagation exclude each other such that the fracture probability remains always low with a maximum at $\Delta_3 \sim F^2$.

In the following sections we will see how the functional dependence of $p_{\rm frac}$ on the system size L governs the crack length distribution whereas the functional dependence of $p_{\rm frac}$ on the external force F governs the critical force for fracture.

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3.5. Crack length distribution

Using the results for the dependence of the fracture probability $p_{\rm frac} = p_{\rm frac}(L)$ on the system size L, we can obtain the distribution $P_{\rm cl}(\ell)$ of arrested crack lengths ℓ using the relation

$$\begin{aligned} P_{\rm cl}(\ell) &= {\rm prob}[Fx + f_{\rm d}(x) > 0 \text{ for } 0 < x < \ell] \times {\rm prob}[F\ell + f_{\rm d}(\ell) < 0] \\ &= p_{\rm frac}(\ell)\, {\rm prob}[F\ell + f_{\rm d}(\ell) < 0], \end{aligned} \tag{40}$$

which expresses the fact that an arrested crack of length ℓ first has to propagate up to this length and then has to be stopped at the length ℓ. Note the nucleation probability $p_{\rm nucl}$ is independent of the sample size L, and $p_{\rm frac}(\ell) = p_{\rm nucl} p_{\rm prop}(\ell)$ only depends via the propagation probability $p_{\rm prop}(\ell)$ on ℓ.

For SRCF the crack length distribution can be calculated from (40) as

$$\begin{aligned} p_{\rm cl}(\ell) &\approx p_{\rm frac}(\ell) \exp\left(-\frac{a}{2\Delta}\ell^{-\delta+2}F^2\right) \\ &\sim \begin{cases} {\rm e}^{-(\ell/x^*)^{2-\delta}} & \text{(SRCF, } \delta < 2\text{)} \\ \left[1 - {\rm e}^{-(x^*/\ell)^{\delta-2}}\right]^{\ell/a} {\rm e}^{-(x^*/\ell)^{\delta-2}} & \text{(SRCF, } \delta > 2\text{).} \end{cases} \end{aligned} \tag{41}$$

For $\delta < 2$, we used the L-independent result (25) for $p_{\rm prop}$ and find that the crack length distribution decays exponentially with the characteristic decay length x^*. Long cracks with $\ell \gg x^*$ are rare because they are unstable and the sample typically fractures instead, For $\delta > 2$, we used (27) for $p_{\rm prop}$ and the crack length distribution decays exponentially with a much smaller characteristic decay length $\sim a$. For $\delta > 2$, it is hard to find macroscopic cracks with $\ell \gg a$ because they are immediately arrested. So in both cases the resulting crack length distribution is exponential but the underlying mechanism giving rise to this result is very different.

Similarly, using (31) for $p_{\rm prop}$, we find for LRCF a crack length distribution

$$p_{\rm cl}(\ell) \sim \left(1 - {\rm e}^{-(\ell/x^*)^{3-\delta}}\right) {\rm e}^{-(\ell/x^*)^{3-\delta}} \qquad \text{(LRCF)}, \tag{42}$$

which decays exponentially for $\delta < 3$ because long cracks are unstable and thus rare, and which decays algebraically for $\delta > 3$ because cracks become arrested before they grow to large lengths.

3.6. Critical force

The information about the critical force which is needed to initiate crack motion at $T = 0$ is contained in the dependence $p_{\rm frac} = p_{\rm frac}(F)$ of the fracture probability on the external force F, which follows from the identity

$$\begin{aligned} p_{\rm frac}(F) &= {\rm prob}[Fx - \gamma + f_{\rm d}(x) > 0 \text{ for all } x] \\ &= {\rm prob}[F > \max_x\{(-f_{\rm d}(x) + \gamma)/x\}] \\ &= {\rm prob}[F > \text{ critical force } F_{\rm c}], \end{aligned} \tag{43}$$

where

$$F_{\rm c} \equiv \max_x\{(-f_{\rm d}(x) + \gamma)/x\} \tag{44}$$

defines the *critical force*. Because the driving force Fx is increasing with crack length x, the combination $f_{\rm d}(x)/x$ determines the critical force. As disorder forces are not bounded from above, the critical force $F_{\rm c}$ might become infinite. The relation (43) shows that the F dependence $p_{\rm frac} = p_{\rm frac}(F)$ of the fracture probability, which we have discussed in previous sections, also contains the information about the cumulative distribution of critical forces for the models (8) or (11) at $T = 0$. The resulting probability distribution of the critical force is given by

$$p(F_{\rm c}) = \partial_F|_{F=F_{\rm c}}\, p_{\rm frac}(F). \tag{45}$$

$F_{\rm c}$ is not yet the critical force for actual fracture, which we will denote by $F_{\rm frac}$ in the following, but $F_{\rm c}$ is the critical force for a single crack with *given initial position*. Once a crack has nucleated at a favorable site and reached the critical length $x_{\rm c}$, it starts to propagate in the direction set by the external loading (eventually its path deviates from a perfect straight line due to disorder; this effect is ignored here). Each crack nucleated at statistically independent nucleation sites experiences a different set of random forces. We define N as the number of such statistically independent nucleation sites. For SRCF it is reasonable to assume that $N \sim (L/x_{\rm c})^2$ assuming that a crack nucleated a distance $x_{\rm c}$ away sees an statistically independent set of random forces. Similar assumptions have been made in random fuse networks [26]–[28]. The result $N \propto L^2$ also holds for LRCF if random energy correlations between cracks at different locations fall off faster then d^{-2} for two cracks a distance d apart. Then the corresponding energy correlation length stays finite. For heterogeneity-induced power-law stress fields decaying as $\sigma \sim r^{-\alpha}$ the random energy correlations only decay for $\alpha > 2$, where we find a power-law decay $\propto d^{4-2\alpha}$. Hence, for LRCF random energy correlations between cracks at different locations fall off faster than d^{-2} only for $\alpha > 3$ or $\delta < -1$. Therefore, $N \propto L^2$ only holds for uncorrelated random forces and impurities. For frozen dislocations, each crack sees essentially the same random forces unless there is a large decorrelation length ξ for the random forces because of another characteristic length scale in the defect distribution. We expect that such a length scale exists in many applications, for example it could be caused by defect clustering effects: in a sample containing grain boundaries with a characteristic distance ξ, energy correlation between frozen dislocations will decay beyond the length scale ξ. If such a length scale exists, we expect $N \sim (L/\xi)^2$ also for LRCF with $\delta > -1$; otherwise $N \sim \mathcal{O}(1)$.

The fracture force $F_{\rm frac}$ is the minimum critical force over all N statistically independent starting points:

$$F_{\rm frac} = \min_N\{F_{\rm c}\}. \tag{46}$$

Therefore, the distribution for the fracture force $F_{\rm frac}$ will attain the limiting form of an appropriate extreme value distribution and differ from the distribution of critical forces $F_{\rm c}$ as given by (45) if N is a macroscopic number. In order to estimate the mean fracture force $\overline{F_{\rm frac}}$, we can use a simple extreme value estimate, according to which the probability to find a critical force $F_{\rm c}$ smaller than the typical minimum value $\overline{F_{\rm frac}}$ should be roughly 1 if N independent samples are taken:

$$p_{\rm frac}(\overline{F_{\rm frac}})N = 1, \tag{47}$$

where the asymptotics of the distribution $p_{\rm frac}(F)$, as given by (38) for SRCF and (39) for LRCF, in the limit of small F is explored.

For SRCF, this estimate leads to

$$\overline{F_{\rm frac}} \sim \begin{cases} F_a (\ln N)^{-(2-\delta)/2} & \text{(SRCF, } 0 < \delta < 2\text{)} \\ F_L \left[\ln \left(L/a \ln N\right)\right]^{1/2} & \text{(SRCF, } \delta > 2\text{).} \end{cases} \tag{48}$$

Thus the mean critical force is essentially given by the two characteristic force scales F_a, see (26), and F_L, see (28), for $\delta < 2$ and $\delta > 2$, respectively. This demonstrates the two different mechanisms leading to fracture for $\delta > 2$ and $\delta < 2$: (i) for $\delta < 2$ fracture is brittle and occurs if the force is sufficient to make a microscopic crack unstable with respect to propagation, i.e. $a \sim x^*$, which happens at $F \sim F_a$. (ii) For $\delta > 2$ fracture is ductile and occurs if the force is sufficient to extend the stable crack of length x^* through the sample of size L, i.e. $L \sim x^*$, which happens at $F \sim F_L$. According to (48) and (28), the average fracture strength *increases* with system size with $F_L \sim L^{(\delta-2)/2}$ for disorder with $\delta \geq 2$, i.e. frozen dislocations, but reduces logarithmically with N for $\delta < 2$. For LRCF the extreme value estimate (47) gives

$$\overline{F_{\rm frac}} \sim F_L N^{-1/2} \qquad \text{(LRCF)}, \tag{49}$$

and the mean critical force is determined by the characteristic force scale F_L, see (32). For $\delta \geq 3$ this force scale grows with the system size and the fracture mechanism is ductile. For $\delta < 3$, on the other hand, F_L becomes small for increasing system size L and a crack can easily propagate after nucleation, see (34), which leads to a vanishing fracture force.

We can go further and calculate the *distribution* of fracture forces $F_{\rm frac}$, i.e. the probability $p(F_{\rm frac})$ to find a critical force $F_{\rm frac}$ for a given realization of disorder, from extreme value statistics. The fracture force $F_{\rm frac} = \min_N\{F_{\rm c}\}$ is the minimum critical force over all N statistically independent starting points and each of the $F_{\rm c}$ is drawn from a distribution $p(F_{\rm c})$ given by (45) with the cumulative distribution $p_{\rm frac}(F)$, as given by (38) for SRCF and (39) for LRCF. Therefore, the distribution of the quantity $1/F_{\rm frac} = \max_N\{1/F_{\rm c}\}$ approaches the type of extreme value distribution which is appropriate for the cumulative distribution of $1/F_{\rm c}$, which is given by

$$\text{prob}[1/F_{\rm c} < x] = 1 - p_{\rm frac}(1/x) \tag{50}$$

according to (43).

For SRCF and $\delta < 2$, the distribution (38) decreases exponentially with $x = 1/F$ such that the corresponding extreme value distribution of $1/F_{\rm frac} = \max_N\{1/F_{\rm c}\}$ is of the *Gumbel type* and

$$\begin{aligned} \text{prob}[F_{\rm frac} > F] &= \text{prob}[1/F_{\rm frac} < 1/F] \\ &\sim \exp\left[-c_1 N^{2/(2-\delta)} \mathrm{e}^{-c_2 (F_a/F) \ln^{\delta/2} N}\right] \qquad \text{(SRCF}, \delta < 2\text{)} \end{aligned} \tag{51}$$

with constants c_1 and c_2. This result applies to random bonds (i) and random impurities (ii) and generalizes previous findings for random fuse models [28], which correspond to the special case of random bonds (i) with $\delta = 0$. Calculating the average value $\overline{F_{\rm frac}}$ with (51) we recover our result (48).

For SRCF and $\delta \geq 2$, the distribution (38) is a power law $(xF_L)^{-2L/a}$ for large $x = 1/F$. Therefore, the corresponding extreme value distribution of $1/F_{\rm frac} = \max_N\{1/F_{\rm c}\}$ is of the *Frechet type* with

$$\text{prob}[F_{\rm frac} > F] \sim \exp\left[-c_1 N (F/F_L)^{2L/a}\right] \qquad \text{(SRCF, } \delta > 2\text{)} \tag{52}$$

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with a constant c_1. Similarly, for LRCF, the distribution (39) is a power law $(xF_L)^{-2}$ for large $x = 1/F$, and the resulting extreme value distribution is of the *Frechet type*

$$\text{prob}[F_{\text{frac}} > F] \sim \exp\left[-c_1 N (F/F_L)^2\right] \qquad \text{(LRCF)}. \tag{53}$$

The probability $\text{prob}[F_{\text{frac}} > F]$ in (51), (52) and (53) equals the probability that the sample will not fracture and, thus, that all N statistically independent cracks are arrested. If $\delta > 2$ for SRCF or $\delta \geq 3$ for LRCF, the resulting fracture probabilities $1 - \text{prob}[F_{\text{frac}} > F]$ in the presence of N statistically independent cracks decrease algebraically for large system sizes L or $F \ll F_L$. Therefore, heterogeneities with long-range forces, such as frozen dislocations (iii) with $\delta = 3$, effectively *arrest* large supercritical cracks, even if small cracks are nucleated with a finite probability.

4. Dynamics of thermally activated crack propagation

While at $T = 0$ any energy barrier leads to crack arrest, thermal fluctuations at $T > 0$ give rise to activated crack propagation even in the presence of local energy barriers. The equation of motion (10) for the crack tip resembles the overdamped motion of a driven particle in a one-dimensional disorder potential $E_{\text{d}}(x)$, which has been studied extensively for disordered systems [29]. At low temperatures, the particle exhibits slow dynamics due to a wide distribution of energy barriers giving rise to anomalously slow diffusion, creep or even particle arrest [30]. In the following, we will focus on the dynamics of crack propagation and neglect the crack surface energy γ in the equation of motion (10).

We consider an ensemble of cracks arrested at $T = 0$ by the random forces and address the question of whether it stays arrested when the finite temperature, $T > 0$, is switched on. To this end we focus now on a *typical* crack and the corresponding disorder-averaged fracture time $\overline{\tau_{\text{frac}}}$, whereas in the previous section we focused on the probability to find a *rare* propagating crack with finite fracture time τ_{frac}. To investigate this problem systematically we start from the Fokker–Planck equation for the probability density $P(x, t)$ corresponding to the equation of motion (10):

$$\partial_t P = -\partial_x J, \tag{54}$$

$$\eta J = -TB^2(x)\partial_x P + B(x)[-\gamma + Fx + f_{\text{d}}(x)]P, \tag{55}$$

where we used the Stratonovich formulation [31].

The stationary solutions $P(x)$ for non-zero constant current J and absorbing boundary condition $P(L) = 0$ are

$$P(x) = \frac{\eta J}{TB^2(x)} \int_x^L \mathrm{d}y \, \exp\left[\int_x^y \mathrm{d}u \, \partial_u \phi\right], \tag{56}$$

which are governed by the effective potential $\phi(x)$ with $\partial_x \phi \equiv (-Fx - f_{\text{d}}(x))/B(x)$. The non-zero constant current J corresponds to the number of cracks the propagation per time and, thus, determines the time to fracture

$$\tau_{\text{frac}} = 1/J. \tag{57}$$

The time to fracture $\tau_{\rm frac}$ can be obtained from the normalization condition $\int_0^L P(x) = 1$:

$$\tau_{\rm frac} \approx \int_0^L {\rm d}x \frac{\eta}{TB^2(x)} \int_0^\infty {\rm d}z\, {\rm e}^{-I(x,z)} \qquad \text{with}$$
$$I(x,z) \equiv \int_x^{x+z} {\rm d}u \left[Fu/TB(u) - \Delta_\delta u^\delta / T^2 B^2(u) \right], \tag{58}$$

where we took the limit of infinite L and performed the disorder average. The behavior of $I(x,z)$ for large z governs the fracture time. Using the asymptotics for large u, $B(u) \approx b/u$, see equation (11), we find

$$I(x,z) \sim Fz^3/3Tb - \Delta_\delta z^{3+\delta}/(3+\delta)T^2 b^2 \tag{59}$$

for large z. In this limit, $I(x,z)$ scales in the same manner for both SRCF and LRCF. Therefore, our results regarding the thermally activated dynamics apply to both types of random forces. Plugging this in (58) we find that the mean fracture time $\overline{\tau_{\rm frac}}$ remains finite for $\delta < 0$, where $I(x,z)$ is large and positive, whereas it diverges for $\delta > 0$, where $I(x,z)$ becomes large and negative. For $\delta = 0$, the mean fracture time is infinite for $\Delta_0 > \Delta_{0,c} \equiv FTb = T\eta c_{\rm R}$, i.e. above the threshold disorder strength $\Delta_{0,c}$, which is independent of the driving force F.

We can also calculate the distribution of random energy barriers, which govern the activated dynamics [32]. As follows from the equation of motion (10), the effective random energy governing thermal activation is $\phi(x)$ with

$$\partial_x \phi \equiv (-Fx - f_{\rm d}(x))/B(x). \tag{60}$$

Therefore, we have to calculate the distribution of barriers $p(E)$ for the random energy $\phi(x)$ of a particle initially at $x = x_i$. This distribution originates from the Gaussian distribution of random forces $f_{\rm d}(x)$ with correlations (4) or (5) and is given by a path integral in 'energy space':

$$p(E) \sim \int_0^L {\rm d}x \int_{\phi(x_i)=0}^{\phi(x_i+x)=E} \mathcal{D}\phi {\rm e}^{-S[\phi(y)]}, \tag{61}$$

$$S \equiv (1/2\Delta_\delta) \int_{x_i}^{x_i+x} {\rm d}y\, y^{-\delta} \left(B(y)\partial_y \phi + Fy \right)^2. \tag{62}$$

The saddle point path $\phi_0(y)$ for the action S obeys the relation $\partial_y[y^{-\delta}(B(y)\partial_y\phi_0 + Fy)] = 0$ which has to be solved with boundary conditions $\phi_0(x_i) = 0$ and $\phi_0(x_i + x) = E$. Focusing on large cracks we consider the limits $x \gg x_i$ and $x \gg b$ and finally find a saddle point action $S_0 \sim (Eb + Fx^3/3)x^{-3-\delta}/\Delta_\delta$. Another saddle point approximation for the x integration in (61) gives a typical barrier size $x_0 \sim (Eb/F)^{1/3}$, which leads to an effective barrier distribution

$$p(E) \sim \frac{1}{E_0} {\rm e}^{-(E/E_0)^{1-\delta/3}} \qquad (\delta < 3) \tag{63}$$

for large cracks ($x \gg x_i$ and $x \gg b$), where

$$E_0 = b^{-1} \Delta_\delta^{3/(3-\delta)} F^{-(3+\delta)/(3-\delta)} \tag{64}$$

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is the characteristic barrier energy. For large cracks with $x \gg x_i$, this distribution scales in the same manner for both SRCF and LRCF. This demonstrates again that our results regarding the thermally activated dynamics apply to both types of random forces. The distribution (63) is a stretched exponential in general, only for $\delta = 0$ it attains a simple exponential form. For $\delta > 3$ large barriers are no longer rare and $p(E)$ cannot be normalized. The corresponding distribution of waiting times τ follows from the Arrhenius law $\tau \sim \tau_0 \mathrm{e}^{E/T}$, where τ_0 is a characteristic microscopic timescale of the dynamics, which depends on η and b:

$$p(\tau) \sim \frac{T}{\tau E_0} \exp\left[-(T/E_0)^{1-\delta/3} \ln^{1-\delta/3} \tau\right] \qquad (\delta < 3). \tag{65}$$

The waiting time distribution assumes a power-law form for $\delta = 0$ and decays faster than a power law for $0 < \delta < 3$. For $\delta > 3$ large waiting times are no longer rare and the distribution cannot be normalized anymore.

Now we consider the crack tip starting at x_i and traveling a distance $x_t \gg x_i$ within a time t. For $0 \leq \delta < 3$, the highest barrier E_t as obtained from the condition $1 = (x_t/x_0) \int_0^{E_t} \mathrm{d}E\, p(E)$ governs the dynamics. This leads to $E_t \sim E_0 \ln^{3/(3-\delta)} x_t$, and using the Arrhenius law $t \sim \tau_0 \mathrm{e}^{E_t/T}$ we find a crack tip dynamics

$$x_t \sim \frac{x_0 E_0}{T} \exp\left[(T/E_0)^{1-\delta/3} \ln^{1-\delta/3} t\right] \qquad (0 \leq \delta < 3). \tag{66}$$

For $\delta = 0$, i.e. a random fracture toughness (i), this represents anomalously slow diffusion with a power-law dynamics $x_t \sim t^{T/E_0}$ where $E_0 = \Delta_0/Fb$. For $T < E_0$ or above the threshold disorder strength $\Delta_{0,c}$ we find a vanishing mean velocity $x_t/t \to 0$ in agreement with our above result of a diverging mean fracture time $\overline{\tau_{\mathrm{frac}}}$. For $\delta < 0$, we find fast brittle fracture $x_t \sim t$ in agreement with our above result of a *finite* mean fracture time. For $\delta > 0$ the crack dynamics is slower than any power law with $x_t/t \to 0$ representing an effective crack arrest in agreement with an infinite mean fracture time. For $\delta \geq 3$, we find complete crack arrest $x_t \approx 0$.

Thus an ensemble of cracks that was arrested at $T = 0$ remains effectively arrested (in the sense of an infinite fracture time or zero average velocity) for heterogeneities with $\delta > 0$, which include frozen dislocations (iii) and random impurities (ii). For random bonds (i) with $\delta = 0$ we find the anomalously slow diffusion with the power-law dynamics. Only for short-range disorder, $\delta < 0$, will thermal fluctuations lead to fast fracture by activation.

5. Statistics of fracture times

The dynamic calculation shows that effective crack arrest with an infinite mean fracture time $\overline{\tau_{\mathrm{frac}}}$ is possible for $\delta \geq 0$ and thus for all three types of disorder (i)–(iii) introduced above. An infinite mean fracture time implies a non-zero probability $p_\infty > 0$ for an infinite fracture time. Thus we find $p_\infty > 0$ for $\delta > 0$ and $p_\infty = 0$ for $\delta > 0$. For $\delta = 0$ we have $p_\infty > 0$ ($p_\infty = 0$) for $\Delta_0 > \Delta_{0,c}$ ($\Delta_0 < \Delta_{0,c}$). Together with our results from the static

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$T = 0$ arguments, we can summarize our findings as

$$\begin{aligned} &p_\infty \approx 1 \qquad \text{for}\, \delta \geq 2\,(\text{SRCF}) \qquad \text{or} \quad \delta \geq 3\,(\text{LRCF}) \quad \text{(iii) frozen dislocat.} \\ &0 < p_\infty < 1 \qquad \text{for}\, 0 < \delta < 2\,(\text{SRCF}) \qquad \text{or} \quad 0 < \delta < 3\,(\text{LRCF}) \quad \text{(ii) random impurit.} \\ &p_\infty = 0 \qquad \text{for}\, \delta < 0. \end{aligned} \tag{67}$$

Random fracture toughness (i) is an example of the special case $\delta = 0$, where we find a power-law distribution of fracture times, $p(\tau_{\text{frac}}) \sim \tau_{\text{frac}}^{E_0/T}$. The tails of the disorder energy barrier distribution are much more effective in crack tip trapping than the finite barriers due to lattice effects [10, 11].

6. Conclusion and discussion

We formulated an equation of motion (10) for the crack tip by incorporating effects from dissipation, thermal fluctuations and frozen heterogeneities into the dynamic fracture criterion $G(x, \dot{x}) = \gamma$. This equation of motion resembles an overdamped Langevin-type equation in a one-dimensional disordered potential. At zero temperature we find that the heterogeneous material can exhibit brittle or ductile fracture, depending on the range of elastic interactions between crack tip and frozen defects. We find different forms of slow crack propagation as summarized in the result (67) for the probability of an infinite fracture time. Frozen defects with $\delta \geq 2$ give rise to complete crack arrest. This applies to hard-worked materials [33] containing frozen dislocations (iii). Also for random fracture toughness (i) and impurities (ii), the disorder potential leads to a slow crack dynamics with zero mean velocity as the crack tip gets trapped in the deep potential minima. This explains the existence of arrested metastable *microcracks* in heterogeneous materials, which can be considerably larger than the critical crack length of the homogeneous material and have been observed in a number of recent experiments [34, 35]. Experimentally observed fracture precursors in heterogeneous materials with power-law waiting time distributions [34]–[37] can also be explained in the framework of our theory as characteristics of the case $\delta = 0$ of random fracture toughness (i), for which we derived a power-law waiting time distribution between crack advance events in equation (65). It remains an issue for future investigations whether the ensemble of arrested microcracks becomes unstable with respect to microcrack coalescence or to slow crack growth by cyclic loading in fatigue experiments.

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