Conservation laws protect dynamic spin correlations from decay: 
Limited role of integrability in the central spin model

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Mazur’s inequality renders statements about persistent correlations possible. We generalize it in a convenient form applicable to any set of linearly independent constants of motion. This approach is used to show rigorously that a fraction of the initial spin correlations persists indefinitely in the isotropic central spin model unless the average coupling vanishes. The central spin model describes a major mechanism of decoherence in a large class of potential realizations of quantum bits. Thus the derived results contribute significantly to the understanding of the preservation of coherence. We will show that persisting quantum correlations are not linked to the integrability of the model but are caused by a finite operator overlap with a finite set of constants of motion.

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Introduction. The two-time correlation function of two observables reveals important information about the dynamics of a system in and out of equilibrium: The noise spectra are obtained from symmetric combinations of correlation functions, whereas the causal antisymmetric combination determines the susceptibilities required for the theory of linear response.

The two-time correlation function only depends on the time difference if at \( t = 0 \) the system of interest is prepared in a stationary state whose density operator commutes with the time-independent Hamiltonian. This is what will be considered in this Rapid Communication. Since correlations generically decay for \( t \rightarrow \infty \), important information about the system dynamics is gained if a nondecaying fraction of correlations prevails at infinite times. Such nondecaying correlations are clearly connected to a limited dynamics in certain subspaces of the Hilbert space. The question arises if such a restricted dynamics is always linked to the integrability of the Hamiltonian. Here integrability means that the Hamiltonian can be diagonalized by a Bethe ansatz which implies that there is an extensive number of constants of motion. Identifying and understanding those nondecaying correlations can be potentially exploited in applications for persistent storage of (quantum) information.

In this Rapid Communication we first prove that persisting correlations are not restricted to integrable systems by using a generalized form of Mazur’s inequality [1,2]. This is in contrast to the behavior of the Drude weight in the frequency-dependent conductivity of one-dimensional systems which appears to vanish abruptly once the integrability is lost, even if only by including an arbitrarily small perturbation. So far, the Drude weight has been the most common application of Mazur’s inequality, see, for instance, Refs. [3–6] and references therein. Second, we apply this approach to the central spin model (CSM) [7] describing the interaction of a single spin, e.g., an electronic spin in a quantum dot [8,9], an effective two-level model for a nitrogen vacancy (NV) center in diamond [10], or a \(^{13}\)C nuclear spin [11], coupled to a bath of surrounding nuclear spins inducing decoherence.

Persisting spin correlations have been found in the CSM by averaging the central spin dynamics over a bath of random classical spins [12,13] or in Markov approximation [14,15]. Finite-size calculations [16,17] of the full quantum problem and stochastic evaluation [18] of the exact Bethe ansatz equations [7] for small system sizes (\( N \leq 48 \)) have also provided evidence for a nondecaying fraction of the central spin correlation, predicting a nonuniversal system-dependent value. Its origin has remained obscure, and it has been speculated that the lack of spin decay might be linked to Bose-Einstein condensatellite physics [18].

Although it is fascinating to identify such nondecaying correlations, it is technically very difficult to rigorously establish them. Approximate methods often miss precisely those intricate aspects allowing correlations to persist, especially when they explicitly exploit the assumption that the system relaxes towards a statistical mixture. Numerical approaches are either restricted in system size [17,19], or they are limited in the maximum time which can reliably be captured [16,17,20]. Even analytical solutions [7] can only often be evaluated in small systems [18]. Thus, a rigorous result establishing the existence of nondecaying correlations is highly desirable, and we resort to Mazur’s inequality for this purpose.

General Derivation. To establish the key idea and to fix the notation we present the following modified derivation related to Suzuki’s derivation in Ref. [2]. We consider the time-independent Hamiltonian \( H \) and the operator \( A \) with a vanishing expectation value \( \langle A \rangle = 0 \) with respect to a stationary density operator \( \rho \), i.e., \( [\rho, H] = 0 \) so that two-time correlation functions only depend on the time difference. Note that \( \rho \) does not need to be the equilibrium density operator. Then, \( \rho \) and \( H \) have a complete common eigenbasis \( |j\rangle \) in a finite-dimensional Hilbert space, and their spectra are \( \rho_j > 0 \) and \( E_j \), respectively. We define the correlation function of \( A \) as

\[
S(t) := \langle A(t) A(0) \rangle = \text{Tr}\{\rho A(t) A(0)\} \quad (1a)
\]

\[
= \sum_{j,m} \rho_j |A_{jm}|^2 \exp[i(E_j - E_m)t], \quad (1b)
\]

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so that Eq. (1b) is its Lehmann representation and $A_{jm} := \langle j | A | m \rangle$ denotes the matrix element of $A$. Physically, $S(t)$ stands for a measurement of $A^\dagger$ at time $t$ after the evolution from the initial state prepared by applying $A$ at $t = 0$. Especially, for $A = S^\dagger$ of a spin $S = 1/2$ in a disordered environment, $S(t)$ is proportional to $\langle S^\dagger(t) \rangle$ if $\langle S^\dagger(0) \rangle = 1/2$, see Supplemental Material for details [21]. If $\lim_{t \to \infty} S(t)$ exists, it is given by

$$S_{\infty} := \sum_{jm} \rho_{jm} |A_{jm}|^2 \delta_{E_j, E_m} \geq 0.$$  \hfill (2)

If $S(t \to \infty)$ does not exist and $|S(t)| < \infty$, the long-time average $\lim_{t \to \infty} \int_0^T S(t) dt = S_{\infty}$ is projecting out the time-independent part $S_{\infty}$ and uniquely defines the nondecaying fraction of the correlation.

In practice, the Lehmann representation (1b) requires the complete diagonalization of $H$, which is not feasible for large systems. Hence one resorts to constants of motion. To this end, we define the scalar product for two operators $X$ and $Y$ as

$$\langle X | Y \rangle := \langle X^\dagger | Y \rangle = \text{Tr}[\rho X^\dagger Y],$$  \hfill (3)

in the super-Hilbert space of the operators. If a set of $M$ conserved linearly independent operators $X_i$ with $[X_i, H] = 0$ is known, one may assume their orthonormality ($X_i | X_m \rangle = \delta_{im}$) provided by a Gram-Schmidt process. Then, we expand the operator of interest $A$,

$$A = \sum_{i=1}^M a_i X_i + R,$$  \hfill (4)

in this incomplete operator basis where $a_i := \langle X_i | A \rangle$ and $R$ is the remaining rest with $\langle X_i | R \rangle = 0 \forall i \in \{1, \ldots, M\}$. Substituting (4) into the definition (1a) yields

$$S(t) = \sum_{i=1}^M |a_i|^2 + S(R)(t),$$  \hfill (5)

with $S^R(t) := \langle R(t) R(0) \rangle$. This relies on the constancy of (i) $\langle X_i^\dagger(t) X_m(0) \rangle = \delta_{im}$, of (ii) $\langle X_i^\dagger(t) R(0) \rangle = 0$, and of (iii) $\langle R(t) X_m(0) \rangle = 0$ all stemming from $[X_j, H] = 0$. For the last relation we have used the cyclic invariance of the trace and $[\rho, H] = 0$.

If we knew $\lim_{t \to \infty} S^R(t) = 0$, we would deduce $S_{\infty} = \sum_{i=1}^M |a_i|^2$. But in general this does not hold because $R$ may still contain a nondecaying part. But (5) implies Mazur’s inequality,

$$S_{\infty} \geq S_{\text{low}} := \sum_{i=1}^M |a_i|^2.$$  \hfill (6)

For a given $H$, the complete set of conserved operators $\Gamma$ is spanned by all pairs of energy-degenerate eigenstates,

$$\Gamma := \{|j \rangle | m \rangle / \sqrt{\rho_m} \text{ with } E_j = E_m\}.$$  \hfill (7)

The elements of $\Gamma$ are orthonormal with respect to the scalar product (3). The coefficient $a_{jm}$ of $X_{jm} = \langle j | m \rangle / \sqrt{\rho_m}$ [22] takes the value $\sqrt{\rho_m} A_{jm}$ so that the right-hand side of (6) equals $S_{\infty}$ as given by the Lehmann representation (2). Thus, the inequality (6) is tight because it becomes exact for the complete set $\Gamma$ of conserved operators. The physical interpretation of Eq. (6) is straightforward in the Heisenberg picture if we view the time-dependent observable $A^\dagger$ as supervector. Its components parallel to conserved quantities (supervector directions) are constant in time because these quantities commute with the Hamiltonian. But all other components, which are perpendicular to the conserved supersubspace, finally decay.

If not all conserved operators are considered, the right-hand side of (6) decreases, and only the inequality holds. Generally, if any subspace of the space spanned by $\Gamma$ is considered, Mazur’s inequality (6) holds. One does not need to know the complete set of eigenstates of $H$ in order to calculate a lower bound: Any finite (sub)set of conserved operators is sufficient.

Now we proceed to generalize Mazur’s inequality for easy-to-use application. Usually, some conserved operators $C_i$ are known, but they are not necessarily orthonormal in general. Rather their overlaps yield a Hermitian positive norm matrix $N$ with matrix elements $N_{ik} := \langle C_i | C_k \rangle$. Each operator $C_i$ can be represented as a linear superposition of the complete set of orthonormal $X_i$’s. These superpositions can be summarized in a matrix $M$ so that $c = M^\dagger x$ where the vectors $x$ and $c$ contain the operators $X_i$ and $C_i$ as coefficients; $M^\dagger$ is the complex (not Hermitian!)$^\dagger$ conjugate of $M$. A short calculation shows that $N = MM^\dagger$.

If we define the vector $a_X$ with complex components $a_i$, the bound $S_{\text{low}}$ can be expressed by $S_{\text{low}} = a_X^\dagger a_X$. In analogy, we compute $a_C$ with complex components $\langle C_i | A \rangle$. Obviously, $a_X = M^{-1} a_C$ holds, and the lower bound is computed by

$$S_{\text{low}} = a_C^\dagger (M^{-1})^\dagger M^{-1} a_C = a_C^\dagger N^{-1} a_C,$$  \hfill (8)

without resorting to orthonormalized operators, relying only on the scalar products of $C_i$ and $A$. We have successfully eliminated the construction of a subset of orthogonal operators $X_i$ and related the lower bound to some known set of linear independent un-normalized conserved operators $C_i$. The general lower bound (8) is our first key result. A possible route to generalizations to various initial states is sketched in the Supplemental Material [21].

**Central Spin Model.** The Hamiltonian of the CSM reads

$$H_0 = \tilde{S}_0 \cdot \sum_{k=1}^N J_k \tilde{S}_k,$$  \hfill (9)

where we assume all spins to be $S = 1/2$ for simplicity. It is a generic model to study the interaction between a two-level system and a bath of spins or more generally a set of subsystems with a finite number of levels. Currently, it is intensively investigated for understanding the decoherence and dephasing in possible realizations of quantum bits [8,9,23,24]. Theoretical tools comprise Chebyshev polynomial technique [17,25], perturbative approaches [15,26,27], generalized master equations [28–30], equations of motion [31], various cluster expansions [32–35], Bethe ansatz [7,18,36,37], density-matrix renormalization [16], and studies of the classical analog [12,13,38,39].

By focusing on $A = S^\dagger_0$, the correlation function defined in (1a) reveals important information on the decay of the central spin. Due to isotropy no other components of the central spin need to be considered. Given the smallness of the
converge and exist for arbitrary numbers of spins because the couplings are bounded from above but become arbitrarily small due to exponential tails of the electron wave function \[8,9,12,23,24\]. This leads to vanishing \(\bar{J}\), implying complete decay for infinite times.

For large but finite times, however, our results include the possibility of slow decays \(S(t) \propto \ln(t)^{-\alpha}\) previously advocated for infinitely large spin baths [13,38,40]. Assuming exponential scaling for the couplings \(J_k \propto \exp(-\beta k)\), where \(\beta\) is inversely proportional to the number of relevant bath spins [41], it is clear that \(J_\infty\) and \(J_\alpha\) converge quickly for \(N \rightarrow \infty\) so that Eq. (11) implies \(S_{\text{low}} \propto 1/N\). Chen et al. [13] have argued that at any given finite time \(t\), only those spins \(\bar{S}_k\) with couplings \(tJ_k \gtrsim 1\) significantly influence the real-time dynamics of the central spin. Hence, only an effective number \(N_{\text{eff}}(t) \propto \ln(t)\) of spins contribute to the correlation function implying \(S(t) \propto 1/\ln(t)\) for such a distribution function.

The lower bound (11) can be improved by considering the three conserved observables \(\bar{I}^z, H_\infty^z,\) and \(I_\infty^z := \bar{I}^z \sum_i I_i^z\). The required vector and matrix elements are given in the Supplemental Material [21]. Still the bound does not exhaust the numerically found value as depicted in the inset of Fig. 1 for \(J_\alpha = 0\) \(J_\alpha\) makes the system nonintegrable, and it will be defined in (12). Even resorting to the integrability of the CSM [7], which implies 0 = [\(H_l, H_\infty\)] with 

\[
H_l := \sum_{k=0,\ldots,\infty}(S_l - e_k)\cdot \bar{S}_k
\]

and \(e_0 = 0, e_k = -1/J_k\), does not account for the full nondecaying fraction obtained in finite-size calculations [17], see the circle in the inset of Fig. 1. The bound has been computed considering \(\bar{I}^z\) and \(H_\infty^z := \bar{I}^z H_l\) for \(l \in \{1, 2, \ldots, N\}\) (for matrix elements see the Supplemental Material [21]).

The above results suggest that the integrability is not the key ingredient for a finite nondecaying fraction. To support this claim we extend the Hamiltonian (9) by adding one extra coupling \(H_0 \rightarrow H\),

\[
H := H_0 + J_\alpha \bar{S}_1 \cdot \bar{S}_N
\]

(12)

between the most weakly and the most strongly coupled bath spins, defined to be at \(k = 1\) and \(N\), respectively. Its value \(J_\alpha\) is

\[\text{CONSERVATION LAWS PROTECT DYNAMIC SPIN . . . PHYSICAL REVIEW B 90, 060301(R) (2014)}\]
chosen to be $O(J_Q)$ so that it constitutes a sizable perturbation even for large spin baths.

The modified time dependence of $S(t)$ is depicted for various values of $J_\alpha$ in Fig. 1. A finite $J_\alpha$ spoils the integrability completely [7] but leaves the quantities $I^z$, $I^z_Q$, and $H^2$ conserved. These three constants of motion, generic for isotropic spin models, are used to obtain the lower bound (red curve) in the inset of Fig. 1. Obviously, $S_{\text{low}}$ is decreased smoothly and only moderately upon increasing $J_\alpha$ in line with the numerically determined $S_{\infty}$. There is no abrupt jump to zero, in contrast to what is known for the Drude weight. The conclusion that integrability is only secondary for the nondecaying spin correlation is our third key result.

At present it remains an open question which conserved quantities one has to include to yield a tight lower bound. We presume that higher powers of $H$, for instance, $I^2 H^2$ have to be considered. Such studies are more tedious and are left for future research. Instead, we take a mathematically less rigorous route based on the estimate by Merkulov et al. [12],

$$S_{\infty} = S_{\text{low}}^{(B)}/[12 S_{\text{low}}^{(B)}(0)],$$

(13)

where $S_{\text{low}}^{(B)}(t)$ is the correlation of the Overhauser field operator $\vec{B}_N := \sum_{j=0}^{N} J_k \vec{S}_j$. Note that an arbitrary $J_\alpha$ can be included because $S_0 \cdot \vec{B}_N$ differs from $H_0$ in (9) only by an irrelevant constant for spin 1/2. This estimate was derived for a classical large Overhauser field [12] and prevails in the thermodynamic limit of the quantum case: The Overhauser field becomes a classical variable upon $N \to \infty$ as shown in Ref. [16].

Thus we now apply the general approach (8) to $A = B_N^0$. Considering only $C_1 = I^z$ as a conserved operator already yields a meaningful lower bound for the Overhauser field correlation function for $N \to \infty$,

$$\frac{S_{\text{low}}^{(B)}}{S_{\text{low}}^{(B)}(0)} = \frac{(J_S + J_0)^2}{(N + 1)(J_Q^0 + J_0^0)}.$$  

(14)

Recall $J_S \propto N$ and $J_0 \propto N$ if the couplings are drawn from a normalized distribution function $p(J)$. This lower bound can be optimized by choosing the arbitrary value $J_0$ such that the bound becomes maximal. With the matrix elements given in the Supplemental Material [21] $S_{\text{low}}^{(B)}$ can be improved considering the three constants $I^z, I^z_Q$, and $H^2$ or all integrals $I^\ell$ and $H^\ell$, $1 \leq \ell \leq N$. The results are also included in Fig. 1 (triangle and square symbols). They hold only for $J_\alpha = 0$ because the estimate (13) applies only in this case. Remarkably, the resulting estimates for $S_{\infty}$ seem to be tight. In particular, the easily evaluated estimate based on all integrals reproduces the numerically found $S_{\infty}$ to its accuracy. We applied the same estimate to the case of $J_0 \propto \exp(-\beta k)$ studied by stochastically evaluating the Bethe ansatz equations and found excellent agreement with the published data with $N \leq 48$ in Ref. [18] as well. Thus we conjecture that the nondecaying fraction $S_{\infty}$ in the central spin model is quantitatively described by $S_{\text{low}}^{(B)}/[12 S_{\text{low}}^{(B)}(0)]$ if $S_{\text{low}}^{(B)}$ is determined from the $N + 1$ integrals $I^\ell$ and $H^\ell$. This constitutes our fourth key result. The small difference, however, between the triangle (from three constants of motion) and the square (from $N + 1$ constants of motion) in Fig. 1 indicates again that the significance of the integrability is limited.

To summarize, four key results are obtained: (i) An easy-to-use version of Mazur’s inequality to prove persisting correlations; (ii) a rigorous finite lower bound for the infinite-time spin correlation in the CSM, valid for the infinite system if the average coupling is finite; (iii) only a small part of the persisting correlation is due to the integrability; (iv) a quantitative estimate for the persisting correlation is conjectured, based on the Overhauser field.

Clearly, the generalized inequality calls for application to other problems [42]. The approach is easy to evaluate and can be used for very large systems and large numbers of constants of motion. Thus it can prove fruitful in the intensely studied field of integrable systems, for instance, in estimating Drude weights. In the context of coherence, in particular, various extensions of the CSM, e.g., by magnetic fields, anisotropies, or more intrabath couplings, suggest themselves to be investigated in the presented manner.

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[22] For clarity, we use a double index here.
[41] In a quantum dot, this is the number of nuclear spins within the localization volume of the electronic wave function, typically $10^4$–$10^6$. It is not the total number of nuclear spins, which is on the order of Avogadro’s constant.